Study of Nonlinear Evolution of Spacetime Fluctuations in Quantum Gravity Inflation

Ken-ji Hamada

https://research.kek.jp/people/hamada/

References

- K. Hamada, Universe 10 (2024) 33 [arXiv:2306.01384]
- K. Hamada, *Quantum Gravity and Cosmology Based on Conformal Field Theory* (Cambridge Scholars Publishing, Newcastle, 2018)

Introduction

How to Enter The Trans-Planckian World

K. H. and F. Sugino, NP B553 (1999) 283
K. H., PTP 108 (2002) 399
K. H., PR D90 (2014) 084038

Why Inflation Is Necessary

The idea of inflation was introduced by Guth, Sato, and Starobinsky to resolve the following problems:

Horizon problem

Why there were larger correlations than horizon size in early universe

Flatness problem

Why curvature remains near zero even after more than 10 billion years

Inflation is also expected to describe the primordial spectrum, which is initial condition of the present universe

Inflation should explain the following feature of the spectrum:

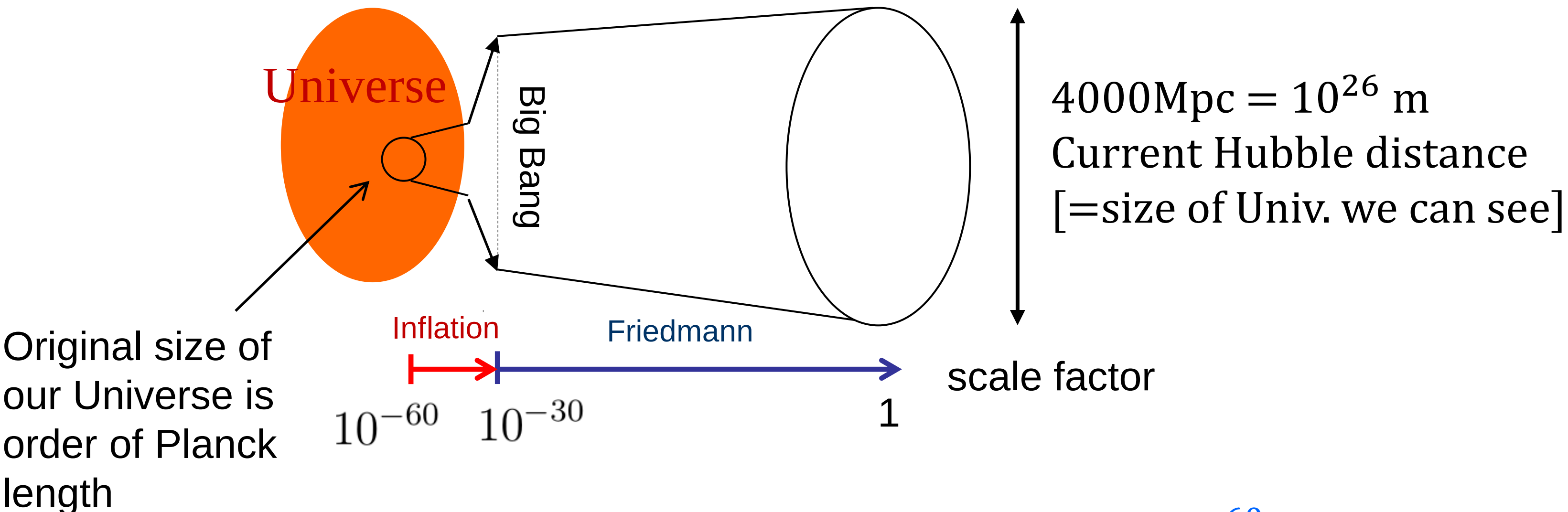
- Why its amplitude is so small
- Why it is almost scale-invariant

What Inflation Suggests

Amazingly, if we believe inflation honestly, then much of Universe we see today was created from a narrower region than Planck length

In a typical inflation scenario, the Universe will be expanded about $10^{60} = 10^{30} \times 10^{30}$ times from its beginning

[Of course, there are scenarios that avoid reaching Planck scale]



Hubble distance = (reduced) Planck length x 10^{60}

Need quantum gravity theory which can go over Planck scale wall *!*

Planck Scale Wall and Renormalizabilility

Planck scale wall:

Einstein gravity has singularities and is not renormalizable

In order to avoid such problems, usually introduce UV cutoff in Planck scale

Some people think of it as an entity of spacetime quantization

However, introducing finite UV cutoff breaks diffeomorphism inv.

Many people believe that there is no world shorter than Planck length, or that such a world is ruled by a physical law other than diffeomorphism inv.

But, this thinking is exactly the root cause of the problem with gravity

singularity, renormalizability, cosmological constant problem, entropy of universe

Bringing UV cutoff to infinity is to make quantum theory of gravity not only diffeomorphism invariant but also renormalizable

Novel Perturbation Method

Inflation gives a hint of how to formulate such quantum theory of gravity

Inflationary spacetime = conformally flat (de Sitter) spacetime

⇩

Quantum spacetime will be described by perturbation expansion around conformally-flat spacetime where Weyl tensor $C_{\mu\nu\lambda\sigma}$ vanishes :

$$g_{\mu\nu} = \underline{\underline{e^{2\phi}}}\,\bar{g}_{\mu\nu}$$

This part is not restricted from $C_{\mu\nu\lambda\sigma} = 0$ condition

➔ treated exactly

$$\bar{g}_{\mu\nu} = \eta_{\mu\lambda}\,(e^{h})^{\lambda}{}_{\nu} = \eta_{\mu\lambda}\left(\delta^{\lambda}{}_{\nu} + h^{\lambda}{}_{\nu} + \frac{1}{2}h^{\lambda}{}_{\sigma}h^{\sigma}{}_{\nu} + \cdots\right)$$

Traceless tensor mode *h* is handled in perturbation by introducing coupling constant later

The most important conformal factor determining distance is treated non-perturbatively

➔ realize background freedom as a special conformal invariance

(whereas conventional perturbation method is defined around flat spacetime of $R_{\mu\nu\lambda\sigma} = 0$)

Quantum Gravity Action

$\mathrm{sgn} = (-1, 1, 1, 1)$

Renormalizable quantum gravity action with conformal invariance in a trans-Planckian region :

conformal in UV

$$I = \int d^4x \sqrt{-g} \left\{ \underbrace{-\frac{1}{t^2} C^2_{\mu\nu\lambda\sigma} - bG_4}_{\text{conformally invariant (no } R^2\text{)}} + \frac{1}{\hbar} \left(\frac{1}{16\pi G} R + \mathcal{L}_{\mathrm{M}} \right) \right\}$$

conformally invariant (no R^2) where G_4 is Euler density

The coupling constant *t* is dimensionless and renormalizable, which controls expansion around $C_{\mu\nu\lambda\sigma} = 0$

(c.f. $-\frac{1}{\mathfrak{g}^2} \mathrm{Tr} F^2_{\mu\nu}$ ⇨ expansion around $F_{\mu\nu} = 0$)

(Note) $\hbar$ appears only before lower actions, because 4-derivative gravity actions are exactly dimensionless, so they contribute only to quantum dynamics, have no classical entity

➔ may be regarded as part of the path integral measure)

Key of Quantization

The action *I* has no 4-derivative dynamics of conformal-factor field ϕ

Unlike conventional perturbation theory around flat spacetime, dynamics (= kinetic + interactions) of ϕ are derived from path integral measure:

$$\int [dg]_g \, e^{iI(g)} = \int [d\phi dh]_{\underline{\underline{\eta}}} \, e^{\underline{iS(\phi,\bar{g})}+iI(g)}$$

diff. inv. measure

practical measures on flat metric so that normal field theory methods can be applied

The *S* (=Jacobian) arises to ensure diffeomorphism invariance, which is Wess-Zumino action for conformal anomaly

[# physical quantity against the name]

Even at UV (*t* = 0) limit, *S* exists, that is Riegert action (= kinetic term of ϕ)

$$S_{\rm R} = \int d^4x \left\{ -\frac{b_1}{(4\pi)^2} \left(2\phi\bar{\Delta}_4\phi + \bar{E}_4\,\phi + \frac{1}{18}\bar{R}^2 \right) \right\}$$

where $E_4 = G_4 - \frac{2}{3}\nabla^2 R$

4-th order conformal inv. differential. op

c.f. Liouville action in 2D QG

Both Singularities and Ghosts are Unphysical

In general, fourth-derivative gravity has the following good properties:

- Renormalizable (coupling constant is dimensionless)
- Action is positive-definite (bounded below) ➔ path integral is stable (unlike EH)
- Singularities can be eliminated as unphysical

  If the action contains Riemann tensor squared, then it diverges for singularities

  Note) Path integral weight for Schwarzschild solution:

  Einstein gravity: $e^{-\int R} = 1$, while 4-deriv. gravity: $e^{-\int R^2_{\mu\nu\lambda\sigma}} = 0$

  statistically forbidden!

Furthermore, this QG theory has the following extra good property:

"BRST" conformal inv. arises as part of diffeomorphism inv. in UV limit

This represents background freedom where all different conformally-flat spacetimes are gauge equivalent, unlike normal conformal inv.

[a realization of Weinberg's asymptotic safety program]

Under this symmetry, all ghost modes become unphysical, not gauge inv., as in the next and appendix

Physical States in A Trans-Planckian World

BRST invariant states

$Q_{\rm B}|{\rm phys}\rangle = 0$ There are an infinite number of physical states

- All ghost modes are unphysical, not BRST invariant
- Physical quantities are given by real composite primary scalars, while tensor quantities are forbidden

consistent with CMB observations

K.H. and S. Horata, PTP 110 (2003) 1169; K.H., PRD 85 (2012) 024028; K.H., PRD 86 (2012) 124006

◆ Ghost modes are unphysical, and have no classical entity like particles

◆ Ghost modes are necessary elements to form the closed BRST conformal algebra, that is, to preserve diff. inv.

◆ Hamiltonian vanishes, but the physical state is not unique, so there is entropy, which is due to the presence of such unphysical ghosts

On Ghosts and Diff. Inv., Further

In renormalizable QG, energy-momentum tensor vanishes as an <u>identity</u>:

$$\int [dg] \frac{\delta}{\delta g_{\mu\nu}(x)} e^{iI} = i\frac{1}{2}\langle \sqrt{-g}\, T^{\mu\nu}(x) \rangle = 0$$

Schwinger-Dyson equation

Then, ghost modes are necessary for Hamiltonian to vanish, i.e., to preserve diffeomorphism invariance

In fact, Einstein gravity also has a ghost due to indefiniteness of Einstein-Hilbert action, but it allows non-trivial solutions such as Friedmann solution

If all modes consist of positive-definite ones, the only state in which total Hamiltonian vanishes is the trivial vacuum ➔ no entropy, no time

Hence, the existence of ghosts itself is not the problem
Ghosts cause problems only when they arise as physical objects

In this QG, ghost modes exist, despite that action is positive-definite, but they all become unphysical under BRST conformal inv.

Thus, we can never see ghosts, but they exist everywhere *!*

What Should Quantum Gravity Show

First of all, we should be aware that fluctuations in gravity represent fluctuations in time and distance

We should be considered that the “measurable” current space and time was born after the fluctuations reduced significantly in early universe

Of course, QG should reveal spacetime with large fluctuations before reducing, which will be realized in a trans-Planckian region

At the same time, QG should provide dynamics in which fluctuations reduce and give rise to the current spacetime

QG is a study of describing changes in state of spacetime, not scatterings of gravitons in a particular spacetime

As said many times, we should move away from the graviton picture*!*

Effective Action, Equations of Motion, and Inflationary Solution

Effective Action -- Weyl Part --

Effective action can be written in terms of running coupling constant

For Weyl sector

$$\Gamma^{\mathrm{W}} = -\left[\frac{1}{t^2} - 2\beta_0\phi + \beta_0 \log\left(\frac{q^2}{\mu^2}\right)\right]\sqrt{-g}C^2_{\mu\nu\lambda\sigma}$$
$$= -\frac{1}{\bar{t}^2(Q)}\sqrt{-g}C^2_{\mu\nu\lambda\sigma}$$

q = momentum measured by flat metric like comoving mom. in cosmology

WZ action of conf. anomaly for Weyl sector, necessary to preserve diff. inv.

Running coupling constant

$$\bar{t}^2(Q) = \frac{1}{\beta_0 \log(Q^2/\Lambda^2_{\mathrm{QG}})}$$

where $Q^2 = g^{\mu\nu}q_\mu q_\nu = \dfrac{q^2}{e^{2\phi}}$

physical momentum squared

New physical scale (= RG inv. $d\Lambda_{\mathrm{QG}}/d\mu = 0$)

$$\Lambda_{\mathrm{QG}} \; (= \mu e^{-1/2\beta_0 t^2})$$

determined from QG inflation scenario

In general, nonlocal corrections are encoded by replacing t^2 in local part with $\bar{t}^2(Q)$

[this holds even in higher loops: K.H., PRD 102 (2020) 125005]

Effective Action -- Riegert Part --

The coefficient of Riegert action, b_1, has quantum corrections, so that

$$\Gamma^{\mathrm{R}} = \int d^4x \left\{ -\frac{b_c}{(4\pi)^2} B \left(2\phi \bar{\Delta}_4 \phi + \bar{E}_4 \phi + \frac{1}{18} \bar{R}^2 \right) \right\} \qquad b_1 = b_c B$$

where $b_c = (N_X + 11N_W/2 + 62N_A)/360 + 769/180$ For Standard Model, $b_c = 7.0$

use this value later

Correction factor *B* is assumed to be summed up in the following

$$B = 1 - \gamma_1 \frac{t^2}{4\pi} + \cdots \to \left[1 + \gamma_1 \frac{\bar{t}^2(Q)}{4\pi} \right]^{-1}$$

(nonlocal corrections are encoded by replacement $t^2 \to \bar{t}^2(Q)$)

This factor, called dynamical factor, expresses that conformal gravity dynamics disappear at dynamical scale :

At $Q = \Lambda_{\mathrm{QG}}$ running coupling diverges, then both Γ^{W} and Γ^{R} vanish

cf. gluon dynamics disappears at QCD scale as

$$\Gamma^{\mathrm{A}} = -\frac{1}{\bar{g}^2(Q)} \mathrm{Tr}(F_{\mu\nu}^2) \to 0 \quad \text{when} \quad \bar{g}^2(Q) \to \infty$$

A Dynamical Model of Spacetime Phase Transition

In order to describe time evolution of the universe, let us first consider homogeneous component of ϕ, denote by $\hat{\phi}$

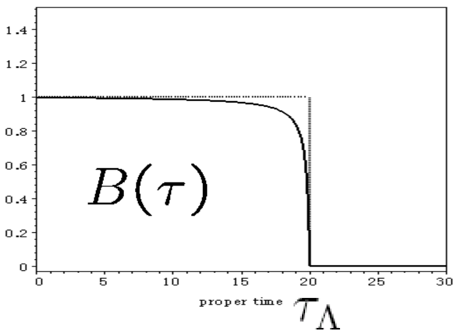
Introduce physical (proper) time $d\tau = a d\eta$, where scale factor is $a = e^{\hat{\phi}}$

Running coupling constant is approximated in time-dependent mean field by making replacement $Q^2 \to 1/\tau^2$ as

$$\vec{t}^2(\tau) = [\beta_0 \log(\tau_\Lambda^2/\tau^2)]^{-1}$$

Running coupling diverges at dynamical time $\tau = 1/\Lambda_{\mathrm{QG}}\ (\equiv \tau_\Lambda)$

Dynamical factor *B* vanishes at dynamical time



In this way, we describe the <u>disappearance</u> of conformal gravity dynamics at phase transition point

Stable Inflationary Solution

$M_{\rm P}\,(=1/\sqrt{8\pi G}) < H_{\rm D} < m_{\rm pl}\,(=1/\sqrt{G})$

Inflationary solution exists when $H_{\rm D} > \Lambda_{\rm QG}$, here their ratio is set as $H_{\rm D}/\Lambda_{\rm QG} = 60$

(why 60 is later)

Homogeneous equation of motion

$$-\frac{b_c}{4\pi^2}\bar{B}(\tau)\partial_\eta^4\hat{\phi} + 6M_{\rm P}^2\, e^{2\hat{\phi}}\left(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right) = 0$$

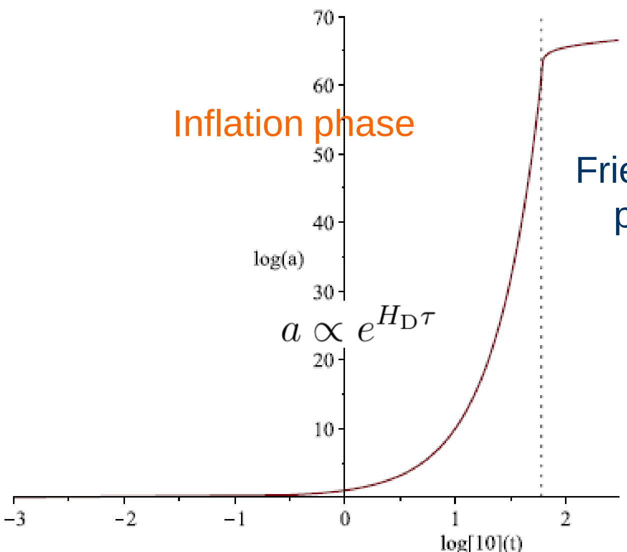
$$H_{\rm D} = \sqrt{\frac{8\pi^2}{b_c}}M_{\rm P} \qquad b_c = 7.0$$

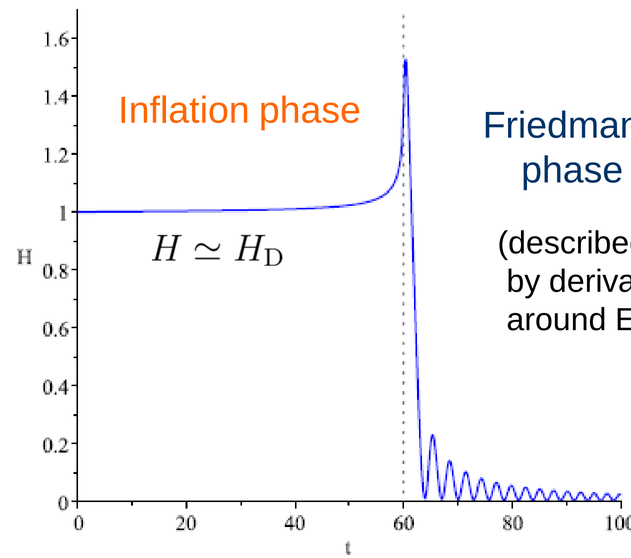
$$\Rightarrow \quad \bar{B}(\tau)\left(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 18H^2\dot{H} + 6H^4\right) - 3H_{\rm D}^2\left(\dot{H} + 2H^2\right) = 0$$

Inflation starts at Planck time ($\tau_{\rm P} = 1/H_{\rm D}$)
and ends at dynamical time ($\tau_\Lambda = 1/\Lambda_{\rm QG}$)

Hubble variable

$$H = \frac{\dot{a}}{a}$$





(described by LEFT defined by derivative expansion around Einstein gravity)

[$H_{\rm D}$ is normalized to unity]

Energy Conservation and Big Bang

Energy conservation equation

matter density

$$\frac{b_c}{8\pi^2}\bar{B}(\tau)\left(2\partial_\eta^3\hat{\phi}\partial_\eta\hat{\phi}-\partial_\eta^2\hat{\phi}\partial_\eta^2\hat{\phi}\right)-3M_{\rm P}^2e^{2\phi}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}+e^{4\hat{\phi}}\rho=0$$

$$\Rightarrow\quad \bar{B}(\tau)\left(2H\ddot{H}-\dot{H}^2+6H^2\dot{H}+3H^4\right)-3H_{\rm D}^2H^2+\frac{8\pi^2}{b_c}\rho=0$$

energy shift

Inflationary solution $H\simeq H_{\rm D}$ indicates initially $\rho\simeq 0$

At transition point $\tau_\Lambda = 1/\Lambda_{\rm QG}$, dynamical factor *B* vanishes and then gravitational energy shifts to matter density ρ, causing big bang

Interactions that create matter density is given by Wess-Zumino actions such as $\phi F_{\mu\nu}^2$

Determination of Dynamical Scale

Number of e-foldings

$$\mathcal{N}_e = \log \frac{a(\tau_\Lambda)}{a(\tau_P)} \simeq \frac{H_D}{\Lambda_{QG}} \qquad \sim 60 - 70$$

Amplitude at transition point is roughly estimated as

$$\left.\frac{\delta R}{\hat{R}}\right|_{\tau_\Lambda} \sim \frac{\Lambda_{QG}^2}{12 H_D^2} \qquad \sim 10^{-4} - 10^{-5}$$ from CMB observation

$$\Lambda_{QG} \simeq 10^{17}\ \mathrm{GeV}$$

(# reduction is shown numerically later)

Comoving dynamical scale

$$\lambda = a(\tau_i)\Lambda_{QG} \qquad \tau_i = 1/E_i \ (E_i \gg H_{\mathbf{D}})$$ some time before inflation begins

can explain sharp falloff at low multipole components of CMB power spectrum:

$$H_0 \simeq \lambda \qquad \lambda \simeq 0.0003\,\mathrm{Mpc}^{-1} \quad \Rightarrow \quad a(\tau_i) \sim 10^{-59}$$ where $a_0 = 1$

Hubble constant

BRST conf. inv. suggests there are no primordial tensor fluctuations involved in CMB
In quantum gravity inflation, tensor-to-scalar ratio does not give limit on inflation scale

Evolution of Universe

correlation length of QG

$\xi_\Lambda = 1/\Lambda_{QG}$

Evolution scenario of universe becomes consistent when

$$\Lambda_{QG} \simeq 10^{17}\ \mathrm{GeV}$$

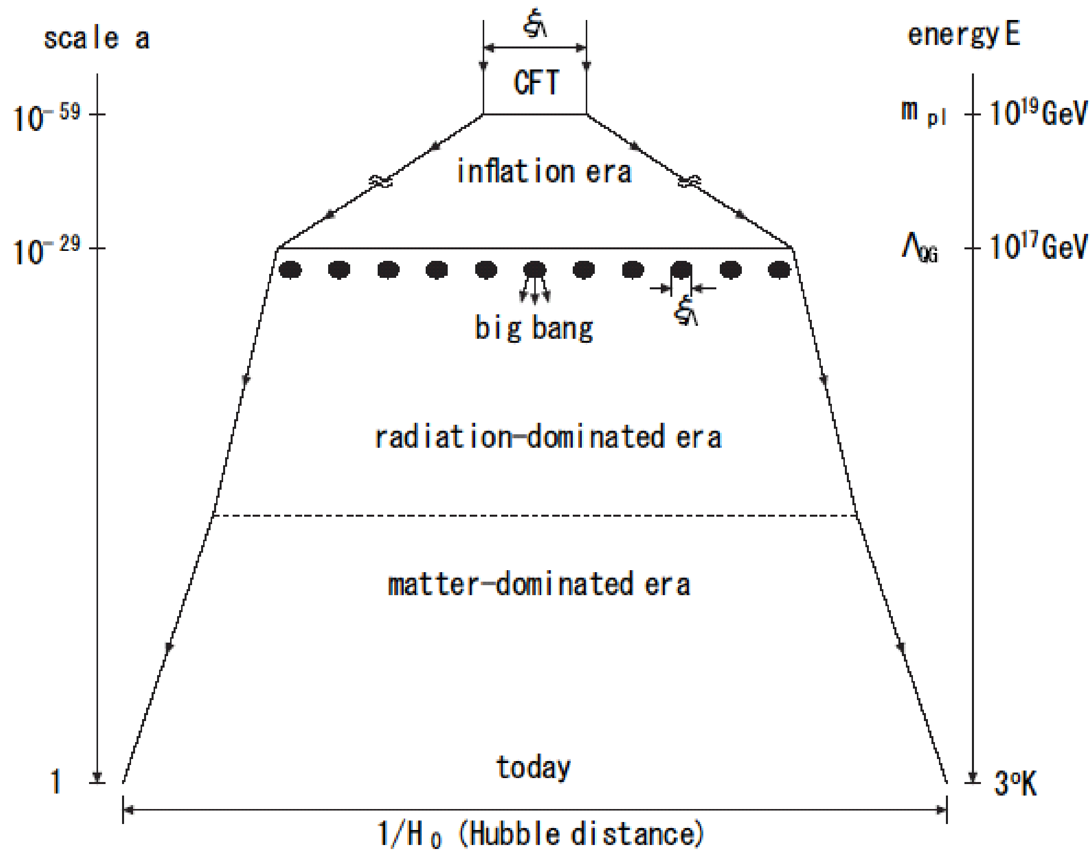
Expansion of the universe 10^{59}

Inflation era

10^{30} $(\Leftrightarrow \mathcal{N}_e = 70)$

Friedmann era

10^{29} $(\Leftrightarrow 10^{17}\mathrm{GeV}/2.7\mathrm{K})$



$1/H_0 \simeq 10^{59}\xi_\Lambda$ (~4000Mpc)

$(\, H_0 \simeq \lambda \,)$

Nonlinear Evolution Equations of Fluctuations in Inflation Era

Equations of Motion for Fluctuations

Derived from variation of the effective action

$$\delta\Gamma = \int d^4x \left(\mathbf{T}^{\mu}{}_{\mu}\delta\phi + \frac{1}{2}\mathbf{T}^{\mu}{}_{\nu}\delta h^{\nu}{}_{\mu} \right) = 0$$

$$\mathbf{T}_{\mu\nu} = \mathbf{T}^{\mathrm{R}}_{\mu\nu} + \mathbf{T}^{\mathrm{W}}_{\mu\nu} + \mathbf{T}^{\mathrm{EH}}_{\mu\nu} + \mathbf{T}^{\mathrm{M}}_{\mu\nu} \qquad \mathbf{T}_{\mu\nu} = \eta_{\mu\lambda}\mathbf{T}^{\lambda}{}_{\nu}$$

Consider scalar fluctuations in conformal-factor field and traceless tensor field

$$\phi(\eta, \mathbf{x}) = \hat{\phi}(\eta) + \varphi(\eta, \mathbf{x})$$

[# homogeneous part already derived]

$$h_{00} = h \qquad h_{ij} = \frac{1}{3}\delta_{ij} h$$

h is treated in linear

(validity is discuss later)

Independent variables are taken as

$$\Phi = \varphi + \frac{1}{6}h \qquad \text{and} \qquad h$$

and ignore $o(h^2)$

where in linear case

$$ds^2 = a^2[-(1+2\Psi)d\eta^2 + (1+2\Phi)d\mathbf{x}^2] \qquad \Psi = \varphi - h/2$$

Trace Equation

$$\mathbf{T}^{\mu}{}_{\mu} = \mathbf{T}^{\mathrm{R}\mu}{}_{\mu} + \mathbf{T}^{\mathrm{EH}\mu}{}_{\mu} = 0$$

where $\mathbf{T}^{\mathrm{W}\mu}{}_{\mu}$ and $\mathbf{T}^{\mathrm{M}\mu}{}_{\mu}$ vanish for linear in *h* due to conformal invariance

Riegert sector

$o(h\Phi)$

$$\mathbf{T}^{\mathrm{R}\mu}{}_{\mu} = -\frac{b_c}{16\pi^2}\bar{B}(4\bar{\Delta}_4\phi + \bar{E}_4) = \mathbf{T}^{\mathrm{R}\mu}{}_{\mu}|_L + \mathbf{T}^{\mathrm{R}\mu}{}_{\mu}|_{NL}$$

Einstein-Hilbert sector

$o(\Phi^n,\, h\Phi^{n-1})$

$$\mathbf{T}^{\mathrm{EH}\mu}{}_{\mu} = M_{\mathrm{P}}^2\sqrt{-g}R = M_{\mathrm{P}}^2 \underline{\underline{e^{2\phi}}}(\bar{R} - 6\bar{\nabla}^2\phi - 6\bar{\nabla}^\mu\phi\bar{\nabla}_\mu\phi) = \sum_{n=1}^{\infty}\mathbf{T}^{\mathrm{EH}\mu}_{(n)}{}_{\mu}$$

where homogeneous part (n=0) is removed

For linear in *h*, trace equation is given by

$$\mathbf{T}^{\mathrm{R}\mu}{}_{\mu}|_L + \mathbf{T}^{\mathrm{R}\mu}{}_{\mu}|_{NL} + \sum_{n=1}^{\infty}\mathbf{T}^{\mathrm{EH}\mu}_{(n)}{}_{\mu} = 0$$

In the following, only *n = 1* and *2* are considered

Trace Equation with $o(\Phi^2, h\Phi)$ Terms

$$
\begin{aligned}
&\frac{b_c}{8\pi^2}\bar{B}(\tau)\Big\{-2\partial_\eta^4\Phi + 4\partial_\eta^2 \not\partial^2\Phi - 2\not\partial^4\Phi - \frac{4}{9}\partial_\eta^2\not\partial^2 h + \frac{4}{9}\not\partial^4 h - 8\partial_\eta^3\hat{\phi}\,\partial_\eta h \\
&-\frac{16}{3}\partial_\eta^2\hat{\phi}\,\partial_\eta^2 h - \frac{4}{3}\partial_\eta\hat{\phi}\,\partial_\eta^3 h + \frac{32}{9}\partial_\eta^2\hat{\phi}\,\not\partial^2 h + \frac{20}{9}\partial_\eta\hat{\phi}\,\partial_\eta\not\partial^2 h
\end{aligned}
$$

Linear of Riegert

$$
\begin{aligned}
&-\frac{16}{3}h\partial_\eta^2\not\partial^2\Phi + \frac{16}{3}h\not\partial^4\Phi - 8\partial_\eta h\partial_\eta^3\Phi + \frac{8}{3}\partial^i h\partial_\eta^2\partial_i\Phi + \frac{8}{3}\partial_\eta h\partial_\eta\not\partial^2\Phi \\
&+\frac{8}{3}\partial^i h\partial_i\not\partial^2\Phi - \frac{16}{3}\partial_\eta^2 h\partial_\eta^2\Phi + \frac{32}{9}\not\partial^2 h\partial_\eta^2\Phi + \frac{16}{9}\not\partial^2 h\not\partial^2\Phi - \frac{4}{3}\partial_\eta^3 h\partial_\eta\Phi \\
&+\frac{20}{9}\partial_\eta\not\partial^2 h\partial_\eta\Phi - \frac{4}{3}\partial_\eta^2\partial^i h\partial_i\Phi + \frac{4}{9}\partial^i\not\partial^2 h\partial_i\Phi\Big\}
\end{aligned}
$$

Nonlinear of Riegert

$o(h\Phi)$

$$
\begin{aligned}
&+M_{\mathrm{P}}^2\, e^{2\hat{\phi}}\Big\{6\partial_\eta^2\Phi + 12\partial_\eta\hat{\phi}\,\partial_\eta\Phi - 6\not\partial^2\Phi + 12\big(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\big)\Phi \\
&+4\partial_\eta\hat{\phi}\,\partial_\eta h + \frac{4}{3}\not\partial^2 h - 8\big(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\big)h
\end{aligned}
$$

Linear (n=1) of EH

$$
\begin{aligned}
&+12\Phi\partial_\eta^2\Phi + 6\partial_\eta\Phi\partial_\eta\Phi + 24\partial_\eta\hat{\phi}\,\Phi\partial_\eta\Phi + 12\big(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\big)\Phi^2 \\
&-12\Phi\not\partial^2\Phi - 6\partial^i\Phi\partial_i\Phi - 8h\partial_\eta^2\Phi + 4\partial_\eta h\partial_\eta\Phi - 16\partial_\eta\hat{\phi}\,h\partial_\eta\Phi + 8\partial_\eta\hat{\phi}\,\partial_\eta h\Phi \\
&-16\big(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\big)h\Phi + 16h\not\partial^2\Phi + 4\partial^i h\partial_i\Phi + \frac{8}{3}\not\partial^2 h\Phi\Big\} = 0
\end{aligned}
$$

Nonlinear (n=2) of EH

$o(\Phi^2,\, h\Phi)$

(when solving eq., time variables are unified to proper time)

Constraint Equation and Its Role

Since there are two fields, trace equation alone is incomplete

Consider another equation

$$\frac{1}{\partial\!\!\!/^2}\left(\mathbf{T}^i{}_i - 3\frac{\partial^i\partial^j}{\partial\!\!\!/^2}\mathbf{T}_{ij}\right) = 0$$

This combination has contributions from Weyl sector, but not from matter sector

Then

$$\frac{16}{3}\frac{1}{\bar{t}^2(\tau)}\left(\partial_\eta^2 h - \frac{1}{3}\partial\!\!\!/^2 h\right) + \frac{b_c}{8\pi^2}\bar{B}(\tau)\left\{\frac{4}{3}\partial_\eta^2\Phi - \frac{4}{3}\partial\!\!\!/^2\Phi + \frac{8}{3}\partial_\eta\hat{\phi}\,\partial_\eta\Phi + 8\partial_\eta^2\hat{\phi}\,\Phi + \frac{8}{27}\partial\!\!\!/^2 h + \frac{8}{9}\partial_\eta\hat{\phi}\,\partial_\eta h + \left(\frac{8}{9}\partial_\eta^2\hat{\phi} - \frac{16}{9}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)h\right\} + M_{\mathrm{P}}\,e^{2\hat{\phi}}\left(-4\Phi + \frac{4}{3}h\right) = 0$$

This equation plays an important role as a constraint for connecting between inflation phase and Friedmann phase :

When $\bar{t}^2(\tau)$ vanishes, *h* vanishes so that fluctuations of $\Phi = \varphi\ (\Phi = \Psi)$ dominates,

while $\bar{t}^2(\tau)$ diverges, fluctuations satisfying $\Phi = h/3\ (\Phi = -\Psi)$ is realized

Friedmann universe

Validity for Treating *h* in Linear

Initial stage of inflation where $\vec{t}^2(\tau)$ is small

Since *h* is small, but Φ is not, consider only nonlinear effects of Φ

Final stage of inflation where $\vec{t}^2(\tau)$ is large

However, as shown later, fluctuations under consideration, related to CMB, decrease during inflation

So, even at this stage, *h* can be treated in linear

This is why constraint equation is handled in linear from beginning to end

Pattern of spectrum considered here is independent of details of strong coupling dynamics (although overall amplitude depends on that)

Spectral pattern is determined in early stage of inflation

How To Handle Nonlinear Terms

Fourier Transform to Comoving Momentum Space

For a dimensionless field *f(x)*, Fourier transform is defined by

$$f(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{k^3} f(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $k = |\mathbf{k}|$

$$\langle f(\mathbf{k}) f(\mathbf{k}')\rangle = \langle |f(\mathbf{k})|^2\rangle k^3 (2\pi)^3 \delta^3(\mathbf{k}+\mathbf{k}')$$

then *f(k)* is also dimensionless

dimensionless power spectrum = $\frac{1}{2\pi^2}\langle |f(\mathbf{k})|^2\rangle$

Fourier transform of a dimensionless field product is defined by

$$\begin{aligned}(fg)(\mathbf{k}) &= k^3 \int d^3\mathbf{x} f(\mathbf{x}) g(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= k^3 \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{p^3 q^3} f(\mathbf{p}) g(\mathbf{q}) e^{i(\mathbf{p}+\mathbf{q}-\mathbf{k})\cdot\mathbf{x}}\end{aligned}$$

Phase factor is expanded using spherical Bessel function as

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(kx) Y^*_{lm}(\Omega_k) Y_{lm}(\Omega_x)$$

In the following, assume that isotropic components of each function contribute predominantly, and thus take

$$f(\mathbf{p}) = f(p) \qquad g(\mathbf{q}) = g(q)$$

Fourier Transform Formulae

Performing angular integrations results in

$$(fg)(k) = \frac{8}{(2\pi)^3} k^3 \int \frac{dp}{p} \frac{dq}{q} f(p) g(q) \int dx x^2 j_0(kx) j_0(px) j_0(qx)$$

where

$$\int_0^\infty dx x^2 j_0(kx) j_0(px) j_0(qx) = \begin{cases} \frac{\pi}{4} \frac{1}{kpq} & \text{for } |p-q| < k < p+q \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$(fg)(k) = \frac{k^2}{(2\pi)^2} \int \frac{dp}{p^2} \frac{dq}{q^2} \theta(k - |p-q|) \theta(p+q-k) f(p) g(q)$$

Similarly, a product with derivatives is given by

$$\begin{aligned}(\partial_i f \partial^i g)(k) &= \frac{8}{(2\pi)^3} k^3 \int dp dq f(p) g(q) \int dx x^2 j_0(kx) j_1(px) j_1(qx) \\ &= \frac{k^2}{(2\pi)^2} \int \frac{dp}{p^2} \frac{dq}{q^2} \theta(k - |p-q|) \theta(p+q-k) \frac{1}{2} (p^2 + q^2 - k^2) f(p) g(q)\end{aligned}$$

Nonlinear Terms in Momentum Space

The $o(h\Phi)$ nonlinear term in Riegert sector

$\left(\lambda = a(\tau_i)\Lambda_{\rm QG} \right.$ gives physical IR cutoff $\left. \right)$

$$\frac{b_c}{8\pi^2}\bar{B}\frac{k^2}{(2\pi)^2}\int_\lambda^\infty \frac{dp}{p^2}\frac{dq}{q^2}\theta(k-|p-q|)\theta(p+q-k)\Big\{-8\partial_\eta h(p)\partial_\eta^3\Phi(q)$$
$$-\frac{16}{3}\partial_\eta^2 h(p)\partial_\eta^2\Phi(q)-\frac{4}{3}\partial_\eta^3 h(p)\partial_\eta\Phi(q)+\left(-\frac{20}{9}p^2+\frac{20}{3}q^2-\frac{4}{3}k^2\right)h(p)\partial_\eta^2\Phi(q)$$
$$-\left(\frac{20}{9}p^2+\frac{8}{3}q^2\right)\partial_\eta h(p)\partial_\eta\Phi(q)+\left(-\frac{2}{3}p^2-\frac{2}{3}q^2+\frac{2}{3}k^2\right)\partial_\eta^2 h(p)\Phi(q)$$
$$+\left[-\frac{2}{9}p^4+4q^4+\frac{2}{9}p^2q^2+\left(\frac{2}{9}p^2+\frac{4}{3}q^2\right)k^2\right]h(p)\Phi(q)\Big\}$$

The $o(\Phi^2,\, h\Phi)$ nonlinear term in Einstein-Hilbert sector

$$M_{\rm P}^2 e^{2\hat{\phi}}\frac{k^2}{(2\pi)^2}\int_\lambda^\infty \frac{dp}{p^2}\frac{dq}{q^2}\theta(k-|p-q|)\theta(p+q-k)\Big\{12\Phi(p)\partial_\eta^2\Phi(q)$$
$$+6\partial_\eta\Phi(p)\partial_\eta\Phi(q)+24\partial_\eta\hat{\phi}\,\Phi(p)\partial_\eta\Phi(q)+12\big(\partial_\eta^2\hat{\phi}+\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\big)\Phi(p)\Phi(q)$$
$$+3\big(p^2+q^2+k^2\big)\Phi(p)\Phi(q)-8h(p)\partial_\eta^2\Phi(q)+4\partial_\eta h(p)\partial_\eta\Phi(q)$$
$$-16\partial_\eta\hat{\phi}\,h(p)\partial_\eta\Phi(q)+8\partial_\eta\hat{\phi}\,\partial_\eta h(p)\Phi(q)-16\big(\partial_\eta^2\hat{\phi}+\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\big)h(p)\Phi(q)$$
$$-\left(\frac{2}{3}p^2+14q^2+2k^2\right)\Big]h(p)\Phi(q)\Big\}$$

Numerical Evaluation By Simplifying Nonlinear Terms

Numerically Calculable System

In order to evaluate the system of nonlinear evolution equations, we have to express the integral as a finite sum

$$\int_{\lambda}^{\infty} dp \to \Sigma$$

So, we have to consider a multiple simultaneous differential equation in which many fields with different momentum are linked

As a first step, consider reducing the number of fields so that we can solve it numerically and quantify nonlinear effects

Moreover, in order to solve the system of evolution equations while preserving the constraint equation, we need to solve it as a boundary value problem (BVP)

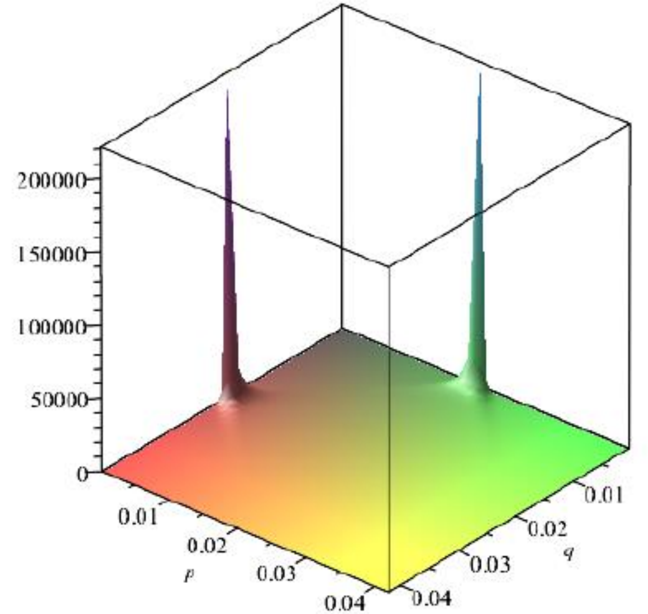
Simplification of Nonlinear Terms

Now notice step-function structure of integrand in nonlinear terms :

$$\frac{1}{(2\pi)^2}\frac{k^2}{p^2q^2}\theta(k-|p-q|)\theta(p+q-k)$$

for physical region $p,\ q\ \geq\ \lambda$

comoving dynamical scale



This function has two peaks at

$$(p,q)\ =\ (\lambda,k)\ \text{and}\ (k,\lambda)$$

Let us simplify nonlinear terms by extracting most contributing parts as follows:

$$(fg)(k)=\frac{1}{(2\pi)^2}\frac{\Delta^2}{\lambda^2}[f(\lambda)g(k)+f(k)g(\lambda)]$$

Δ is a new phenomenological parameter

⇨ reduce multi-line system to two-line system

Parameters and Boundary Conditions

From inflationary scenario discussed before

Ratio of two scales is set as $\frac{H_{\rm D}}{\Lambda_{\rm QG}} = \frac{m}{\lambda} = 60$ here, normalized to $H_{\rm D} = 1$

Comoving Planck scale is taken as

$$m = a(\tau_i) H_{\rm D} = 0.02\,{\rm Mpc}^{-1}$$

Comoving dynamical scale is then

$$\lambda = a(\tau_i)\Lambda_{\rm QG} = 0.00033\,{\rm Mpc}^{-1}$$

Initial time is set as $\tau_i = 1/E_i = 10^{-3}$ far before Planck time $\tau_{\rm P} = 1$,

then transition time is $\tau_\Lambda = 1/\Lambda_{\rm QG} = 60$

momentum dep.

$$\frac{k^2}{H_{\rm D}^2 a(\tau)^2} = \frac{k^2}{m^2 \bar{a}(\tau)^2}$$ with $\bar{a}(\tau_i) = 1$

Initial and boundary values of fluctuations

$$\langle \varphi(\tau_i, \mathbf{x})\varphi(\tau_i, \mathbf{y})\rangle = -\frac{1}{4b_c}\log(\mathbf{x}-\mathbf{y})^2 \xrightarrow[\text{Fourier transf.}]{} \langle|\varphi(\tau_i, k)|^2\rangle = \frac{\pi^2}{b_c}$$

: initial power spectrum is scale invariant

⇨ $\Phi(\tau_i, k) = \frac{\pi}{\sqrt{b_c}}$ for $k \geq \lambda$ and derivatives of Φ up to three times vanish

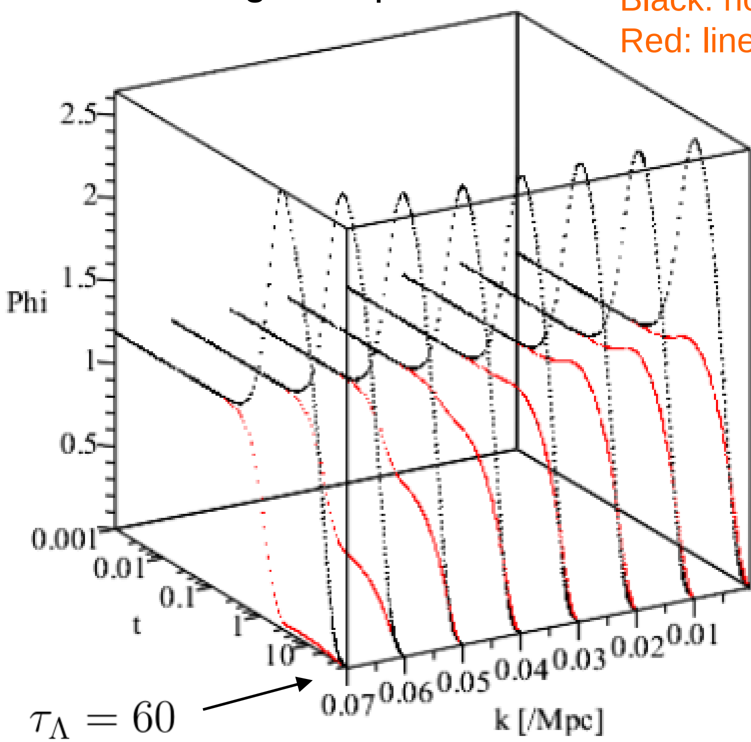
$h(\tau_i, k) = 0$ and $h(\underline{\tau_\Lambda}, k) = 3\Phi(\underline{\tau_\Lambda}, k)$ (←required from constraint eq.)

Numerical Results (Reduction of fluctuations)

Time Evolution of Φ

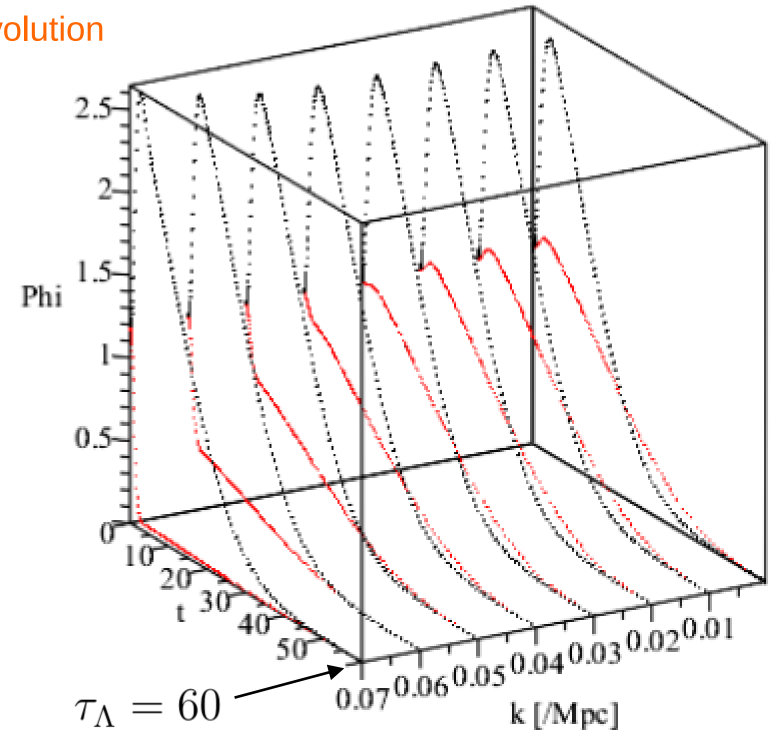
Black: nonlinear evolution
Red: linear evolution

Log-time plot



$\tau_\Lambda = 60$

Normal-time plot



$\tau_\Lambda = 60$

Comoving scales: $m = 0.02\,\mathrm{Mpc}^{-1}$

$\lambda\,(= m/60) = 0.00033\,\mathrm{Mpc}^{-1}$

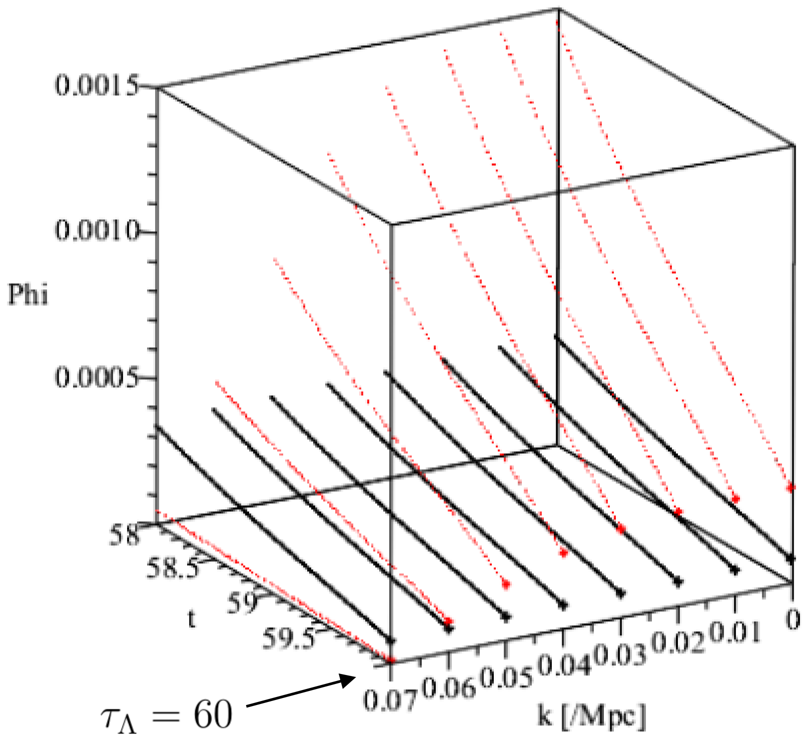
Parameters that control strength of nonlinear terms: $b_c = 7$

$\Delta/2\pi = 1.3\lambda$

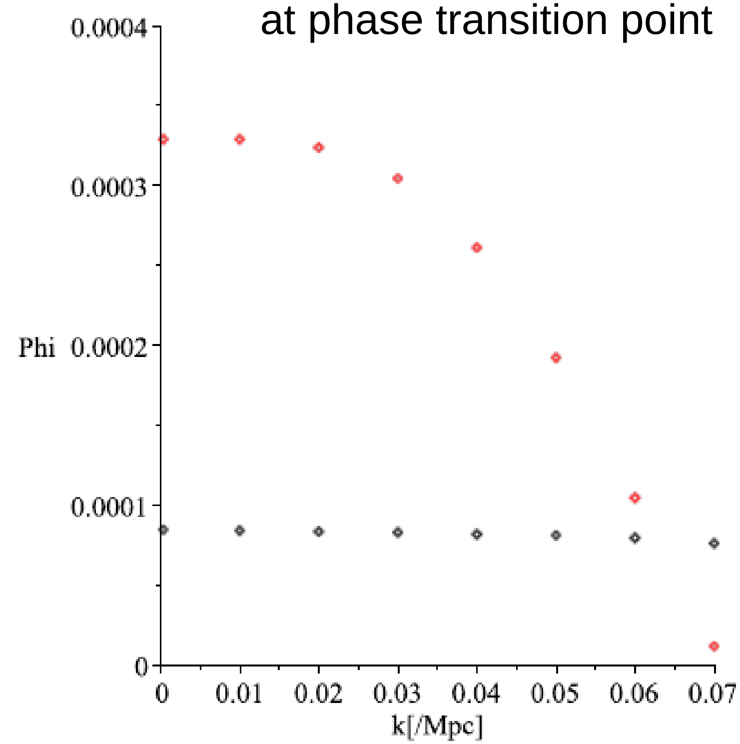
Final overall amplitude is adjusted by dynamical parameters β_0 and γ_1, but spectral pattern is indep. of them

Primordial Spectrum

Last stage of evolution



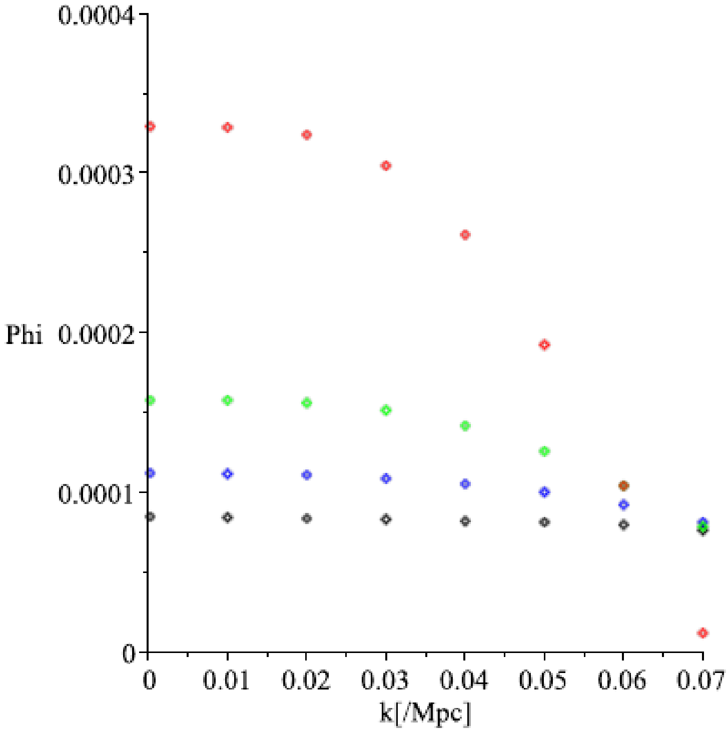
Primordial spectrum
at phase transition point

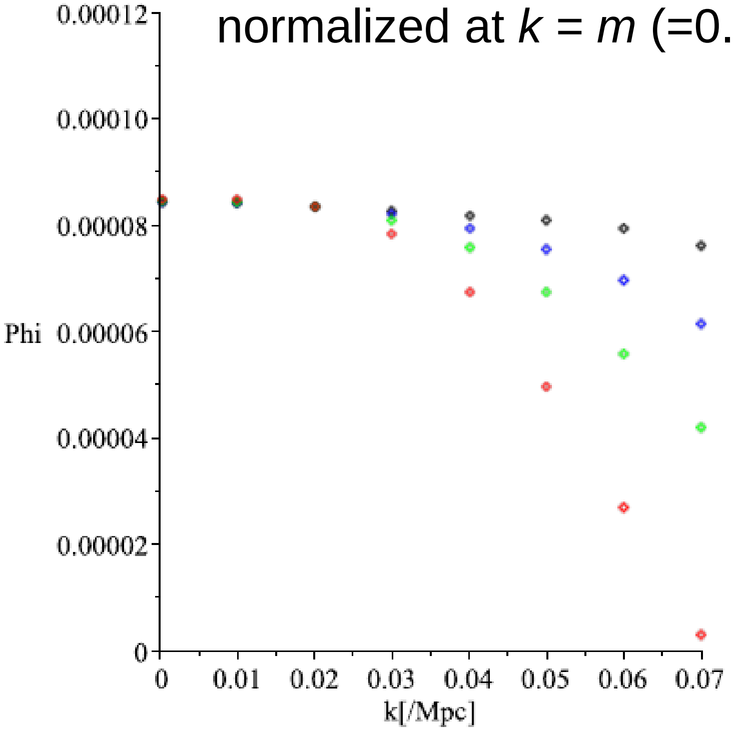


Nonlinear terms work to maintain initial scale-invariance beyond comoving Planck scale $m = 0.02\,\mathrm{Mpc}^{-1}$

higher momentum region may need higher-order nonlinear terms

Results for Various Delta





normalized at $k = m$ (=0.02)

$$\frac{\Delta}{2\pi\lambda} = 1.3,\ 1,\ 0.7$$

Conclusion

Quantum gravity inflation suggests that observed CMB anisotropy spectra contain real quantum gravity effects

- I determined and examined various nonlinear terms, such as exponential factor of conformal mode, that contribute to early stage of inflation with still large amplitude

  Running coupling constant is then expressed by time-dependent average in the spirit of mean-field approximation

- I showed that spacetime fluctuations reduces in amplitude during inflation

  This implies that the universe becomes real world where time and distance can be measured accurately ➔ graviton picture becomes applicable

- I confirmed that nonlinear terms have expected effects by which initial scale-invariance is maintained until phase transition point up to relatively high-momentum regions

  But, incomplete yet due to simplifications and ignoring third and more terms

  To what extent will such an effect be maintained? Could unexpected structures emerges in regions of even higher momentum? ➔ future tasks

This is a challenge toward derivation of precise primordial spectra, and I expect that it will lead to resolution of cosmological tensions in the future

Calculation Device

In order to handle BVP for differential equations where some variables that are not observable diverge at boundaries, special programs are needed

W. Enright and P. Muir, Runge-Kutta software with defect control for boundary value ODEs, SIAM J. Sci Comput. 17 (1996) 479

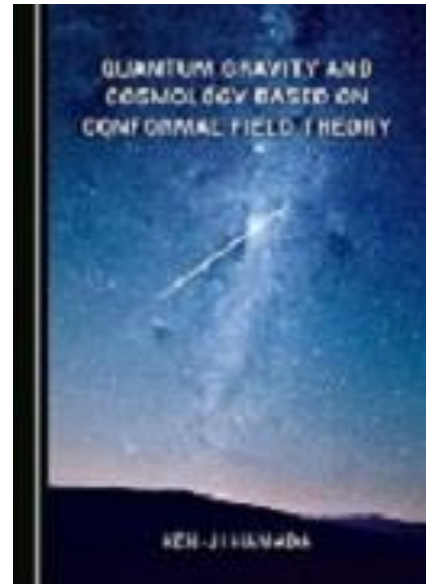
Moreover, here need to solve simultaneous equation with multiple variables

Maple software, not Mathematica, has a built-in program for such BVP, and use it here

In order to perform large-scale computations involving many fields with different momentums with higher precision, special program such as Fortran software, BVP_SOLVER, is required

See Books For Mathematical Details

共形場理論を基礎にもつ
量子重力理論と宇宙論
(プレアデス出版、2016）



Quantum Gravity and Cosmology
Based on Conformal Field Theory
(Cambridge Scholars Publishing, 2018)

APPENDIX

Background Freedom as BRST Conformal Inv.

Background freedom arises in UV limit of $t \rightarrow 0$ as part of diffeomorphism invariance $\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\nu\lambda}\nabla_\mu \xi^\lambda$, in which ξ^λ is given by conformal Killing vectors c^λ :

[other gauge d.o.f. are fixed, e.g. in radiation gauge]

$$\delta_{\rm B}\phi = c^\mu \partial_\mu \phi + \frac{1}{4}\partial_\mu c^\mu$$

$$\delta_{\rm B} h_{\mu\nu} = c^\lambda \partial_\lambda h_{\mu\nu} + \frac{1}{2} h_{\mu\lambda}\left(\partial_\nu c^\lambda - \partial^\lambda c_\nu\right) + \frac{1}{2} h_{\nu\lambda}\left(\partial_\mu c^\lambda - \partial^\lambda c_\mu\right)$$

This conformal symmetry is a gauge symmetry, not a normal one

All theories with different backgrounds connected to each other by conformal transformations are gauge equivalence

||

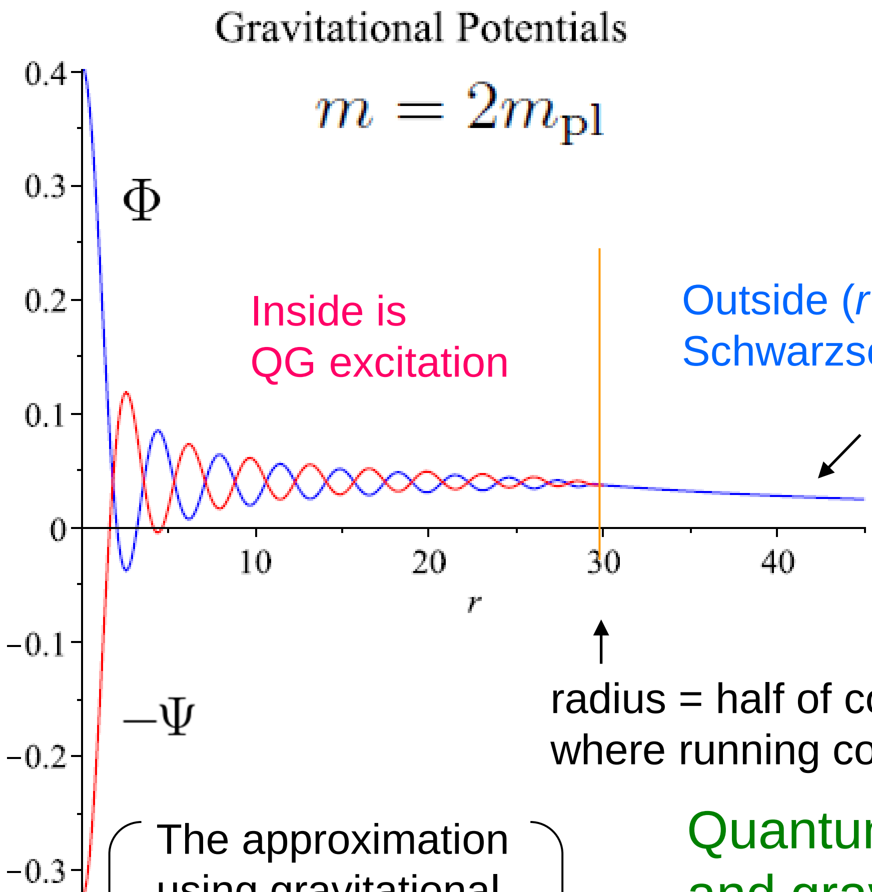
Independence of how to choose background metric

$$\hat{g}_{\mu\nu} \cong e^{2\omega(x)}\hat{g}_{\mu\nu}$$

[That for tensor mode is less dominant]

In other words, since $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$, a conformal change of $\hat{g}_{\mu\nu}$ can be absorbed by a shift change of ϕ, while ϕ is an integration variable and its measure is invariant under the shift so that the theory does not change ← performing integration is essential

Spherical and Static Excitation



A solution of $T_{\mu\nu} = 0$ with $T^{\rm M}_{\mu\nu} = 0$ solved under approximation that VEV of running coupling constant is replaced with a position-dependent mean field

K. H., Phys. Rev. D 102 (2020) 026024

Inside is QG excitation

Outside (r > 30) is Schwarzschild solution

$$\Phi = -\Psi = \frac{r_g}{2r} \qquad r_g = 2Gm \, (= 4/m_{\rm pl})$$

$$m = \int_{|x| \le \frac{\xi_\Lambda}{2}} d^3\mathbf{x} \, T^{(4)}_{00}(\mathbf{x})$$

radius = half of correlation length $\xi_\Lambda \, (= 1/\Lambda_{\rm QG}) \simeq 100 \times l_{\rm pl}$ where running coupling diverges

The approximation using gravitational potentials is valid only for $r_g/2\xi_\Lambda \ll 1$

Quantum gravity is activated inside excitation and gravitational fields oscillate greatly